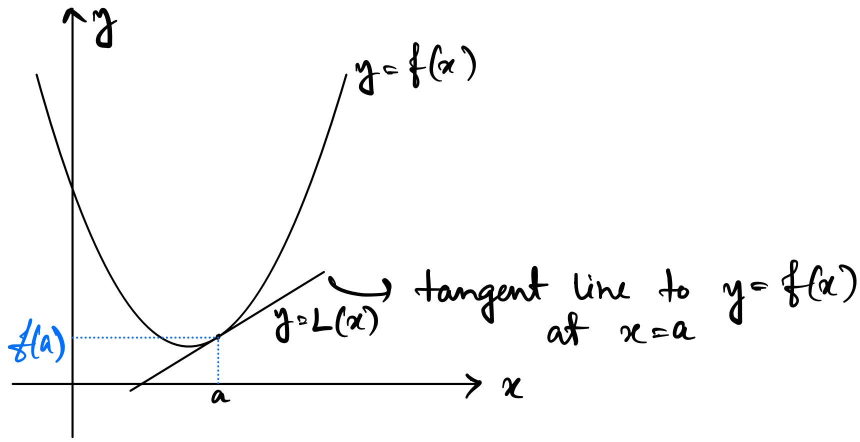
Linear Approximations & Differentials:



⊛ **Note**: Tangent is the best straight line approximation to $y = f(x)$ at $x = a$.

Now the equation of the tangent line passing through $(a, f(a))$ with slope $f'(a)$ is given by

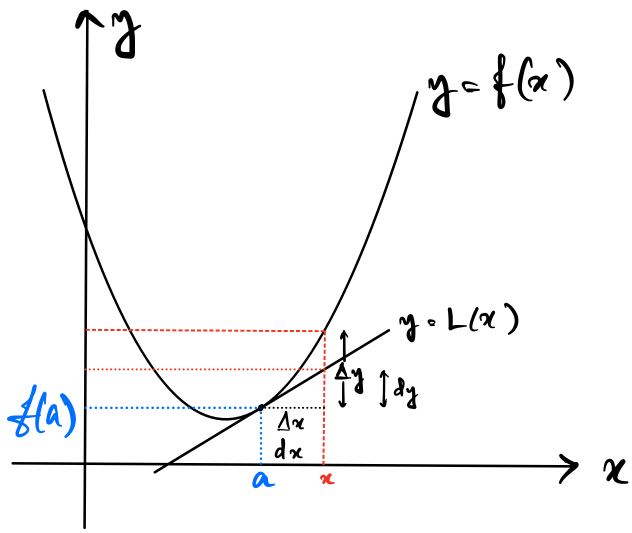
$$y - f(a) = f'(a)(x-a)$$

$$\Rightarrow y = \underbrace{f(a) + f'(a)(x-a)}_{L(x)},$$

the linear approximation.

Eg. (a) Find the linear approximation of $f(x) = \sqrt[3]{\sin(e^2 x)}$ near $x = \frac{1}{e^2}$, given $\frac{1}{e^2} = 0.13533$

(b) Use linear approximation to estimate $\sqrt[3]{\sin\left(\frac{e^2}{10}\right)}$



Arbitrary Distance from a

$$\Delta x = x - a$$
$$\Delta y = f(x) - f(a)$$

$$\boxed{\Delta y = f(x+\Delta x) - f(x)}$$

Infinitesimal Distance from a.

$$dx = x - a$$
$$\begin{aligned} dy &= L(x) - L(a) \\ &= [f(a) + f'(a)(x-a)] - f(a) \\ &= f'(a)(x-a) \\ &= f'(a)\,dx \end{aligned}$$

$$\boxed{dy = f'(x)\,dx} \longrightarrow \text{Differentials.}$$

Eg. Let $y = x^2 + 7$

(a) Find dy in terms of dx.

(b) Approximate Δy, if $x = 1$ & $\Delta x = 0.001$

$dy = f'(x)\,dx$ & $\Delta y = f(x+\Delta x) - f(x)$.

(a) $f(x) = x^2 + 7 \Rightarrow f'(x) = 2x$

So, $dy = 2x\,dx$.

(b) $\Delta y = f(1 + 0.001) - f(1)$

$= f(1.001) - f(1)$

$= [(1.001)^2 + 7] - [1^2 + 7]$

$= (1.001)^2 - 1^2 = (1.001 + 1)(1.001 - 1)$

$= (2.001)(0.001)$

$= 0.002001$